The many primitives of mereology

Josh Parsons

December 21, 2012

Abstract

Formal mereologies are axiomatised in a variety of different ways, with a variety of different primitives. In this paper I distinguish three such ways, and show that not every way is suitable for every mereology.

1 Introduction

Classical Extensional Mereology (CEM) has an interesting feature: it admits of various different axiomatisations in terms of various different primitives. Some, like Eberle, treat the "is a (proper or improper) part of" predicate as the primitive; some, like Simons, treat the "is a proper part of" as the primitive; some, like Goodman, treat the "overlaps" predicate (or its negation, "is disjoint from") as the primitive.[1]

This seems to me to be a metaphysically significant feature of CEM. If CEM is correct — if all its theorems are true, then metaphysicians have a choice to make in how we understand the mereological nature of the world. We may think of the mereological relation either as a relation of part to whole, or as a relation of overlap; for if we give a metaphysical theory about one, we thereby give a metaphysical theory about the other. We may choose which we think of as more metaphysically fundamental, for the they are interdefinable. However, if CEM is not correct, then perhaps we do not have this choice. Perhaps part-whole cannot

[1] For Eberle, see the discussion in Simons (1987, pp. 50-52); for Simons, see his (1987, pp. 25–41); for Goodman, see his (1951, pp. 42-51) and the discussion in Simons (1987, pp. 48-50). It should be noted that axiomatisations may also differ on what axioms are used, as well as on what the primitive is. For recent work on alternative axiomatisations of CEM of this kind, see Hovda (2009).

be defined in terms of overlap; in which case we must choose part-whole as the metaphysical fundamental mereological relation (if any relation is).

It would be nice to know, given a mereological theory, which styles of axiomatisation are possible for it. That way, if CEM is not correct, but some weaker mereology is, we could know whether that mereology would let us regard overlap, or part-whole, or proper part, as the fundamental mereological relation.

In this paper, I explore which mereologies are axiomatisable in which of the three styles — Eberle, Simons, and Goodman — described above. I claim that all mereologies must have an Eberle style axiomatisation: if a theory does not permit the definition of its vocabulary in terms of "is part of", it is not a mereology! I then prove that all and only those mereologies that feature the strong supplementation principle have an axiomatisation in Goodman's style; and that (with some caveats) all mereologies have an axiomatisation in Simons's style.

The formal notation used in this paper is intended to promote readability and avoid distracting logical issues. The logic is classical first-order predicate calculus with identity. Open sentences should be understood as implicitly universally quantified (unless otherwise stated). Where scope is left ambiguous, negation takes narrow scope; conditionals and biconditionals take wide scope. Detailed proofs of some classical sequents are set out in the appendix — where the text asserts that some principle is a logical consequence of some others, the proof may be found there.

2 Goodman-axiomatisation and (SSP)

Goodman's axiomatisation of CEM treats overlap ($\circ$) as primitive, and defines "is a (proper or improper) part of" ($<$) in the following way:

$$x < y \equiv_{df} (\forall z)(z \circ x \to z \circ y) \tag{Def<}$$

Let us say that a theory is *Goodman-axiomatisable* iff it has an axiomatisation treating $\circ$ as primitive, and defining $<$ using (Def<). A theory is Goodman-axiomatisable iff (Def<) is an admissible rule in it — iff all instances of (Def<) are provable in the theory. In classical first-order logic, a definition like (Def<) is admissible in a theory iff the corresponding biconditional is a theorem of that theory. So, a theory is Goodman-axiomatisable iff the principle (GA) is a theorem of it:

$$x < y \leftrightarrow (\forall z)(z \circ x \to z \circ y) \tag{GA}$$

Eberle's axiomatisation of CEM, in contrast, treats $<$ as primitive, and defines $\circ$ as follows:

$$x \circ y \equiv_{df} (\exists z)(z < x \wedge z < y) \tag{Def$\circ$}$$

We may say that a theory is *Eberle-axiomatisable* iff (Def$\circ$) is admissible in it; that is, iff the principle (EA) is a theorem of it.

$$x \circ y \leftrightarrow (\exists z)(z < x \wedge z < y) \tag{EA}$$

Every mereology whatever should be Eberle-axiomatisable. If a mereological theory is already axiomatised in Eberle's way, then of course it is. If a theory is axiomatised in any other way, then if (EA) is not a theorem, then something is very wrong with it: $\circ$ does not mean "overlap" in the sense in which mereological overlap is normally informally explained.

Not every mereology whatever is Goodman-axiomatisable, however — for some weak mereological-looking theories do not have (GA) as a theorem.[2] More, however, can be said. I will show that a mereology is Goodman-axiomatisable iff it has as a theorem a principle called the *strong supplementation principle*, or (SSP):

$$\neg x < y \rightarrow (\exists z)(z < x \wedge \neg z \circ y) \tag{SSP}$$

(SSP) is a much-discussed and controversial theorem of classical mereology. Non-classical mereologies often explicitly affirm or reject it. That makes it easy to tell which non-classical mereologies are Goodman-axiomatisable. Those mereologies that reject (SSP) are not presented using a Goodman-style axiomatisation — my argument shows that they *cannot* be.

Obviously there are formal systems with symbols that look like $<$ and $\circ$ of which (SSP) is not a theorem and of which $\circ$ is the primitive. Those systems, I would claim, are not mereological — in them, $\circ$ doesn't have the right formal characteristics to be capable of meaning "mereological overlap". Since it is a controversial matter just what a theory must do to count as a mereology, I define a class of theories, the *pre-mereologies*, of which the mereologies are, uncontroversially, a subclass. My claim is that all and only those pre-mereologies that have (SSP) as a theorem are Goodman-axiomatisable. Since the mereologies are among the pre-mereologies, all and only those mereologies that have (SSP) as a theorem are Goodman-axiomatisable.

[2]For example, the system Simons describes as SA0-3 (1987, p. 28) is, perhaps, a kind of non-extensional mereology, but is not Goodman-axiomatisable. The four-element model shown on p. 28 of Simons's book is a countermodel to (GA).

Let a *pre-mereology* be any first order theory with at least two dyadic predicates, $<$ and $\circ$, and at least the following theorems:

$$x < y \wedge y < z \rightarrow x < z \tag{Trans}$$

$$x < x \tag{Refl}$$

$$x \circ y \leftrightarrow (\exists z)(z < x \wedge z < y) \tag{EA}$$

To be a pre-mereology, that is, a theory must treat $<$ as a pre-ordering, and must be Eberle-axiomatisable. It's reasonable I think, to expect that all mereologies will have these features. So every mereology is a pre-mereology; whatever holds true of all pre-mereologies holds true of all mereologies.

If a pre-mereology is Goodman-axiomatisable, then it has (SSP) as a theorem. *Proof:* if a pre-mereology is Goodman-axiomatisable, then it has (GA) as a theorem. But (SSP) is a consequence of (GA), (Refl) and (EA) (this, and other, first order proofs are set out in the appendix). (Refl) and (EA) are theorems of every pre-mereology, so (SSP) is a theorem of every Goodman-axiomatisable pre-mereology.

If a pre-mereology has (SSP) as a theorem, then it is Goodman-axiomatisable. *Proof:* (GA) is a consequence of (SSP), (Trans), (Refl), and (EA). (Trans), (Refl), and (EA) are theorems of every pre-mereology, so every pre-mereology that has (SSP) as a theorem also has (GA). If a pre-mereology has (GA) as a theorem, then it is Goodman-axiomatisable, so every pre-mereology that has (SSP) is Goodman-axiomatisable.

So, a pre-mereology has (SSP) iff it is Goodman-axiomatisable; all mereologies are pre-mereologies; therefore, all and only mereologies with (SSP) are Goodman-axiomatisable. This sheds useful light on the significance of (SSP): it can thought of as a kind of supervenience principle — no two things can differ as regards what parts they have (or what they are parts of) without differing as regards what they overlap.

3 Simons-axiomatisation

A third style of axiomatisation of classical mereology is used by Simons. He takes "is a proper part of" as his primitive, and defines "is part of" as "is a proper part of or identical to". Before I introduce any notation here, I would like to clear up some terminological problems. "proper part" appears to be used in two senses in the literature. The first sense is the one suggested by Simons's definition, that "proper part" means "part of but not identical to"; so that "part of" may be adequately defined in Simons's way. I call this relation "non-identical part", and use the symbol $\lneq$ for it. In a theory in which it is not a primitive, $\lneq$ may be defined thus:

$$x \lneq y \equiv_{df} x < y \wedge y \neq x \tag{Def$\lneq$}$$

The second sense of "proper part" may be glossed as follows. Some weak mereologies allow the part-whole relation to fail to be anti-symmetric — allowing two things to be parts of each other. Let us say that x and y are *mutual parts* iff x is part of y and y is part of x.[3] Note that each thing is mutually part of itself (even if part-whole is anti-symmetric); so another way of defining "proper part" in such a way that nothing ever counts as a proper part of itself is to say that "proper part" means "part of but not mutually part of". I will call this relation "non-mutual part", and use the symbol $\ll$ for it. Assuming (Refl) and (Trans), x and y have all and only the same parts iff they are parts of each other. So in a mereology in which it is not a primitive, $\ll$ may be defined thus:

$$x \ll y \equiv_{df} x < y \wedge \neg y < x \tag{Def$\ll$}$$

How are we to tell, given a mereology in which some symbol is identified as "proper part", which of these two relations is meant? It's easy if "proper part" is a defined relation, for then we can look at which of the two definitions above introduced it. But what if we are looking at a Simons-style axiom system where "proper part" is the primitive? Then I suggest we should look at whether the biconditionals corresponding to the definitions above are theorems of the system. If the biconditional $x * y \leftrightarrow x < y \wedge x \neq y$ is a theorem, then the relation written $*$ is non-identical part. If the biconditional $x * y \leftrightarrow x < y \wedge \neg y < x$ is a theorem, then the relation written $*$ is non-mutual part.[4]

A characteristic of the extensional theories that Simons describes in this way is that both of the biconditionals above are theorems. Theories like that are in effect saying that x is a non-mutual part of y iff x is a non-identical part of y; which is to say that x and y are mutual parts iff x and y are identical. In these systems it

[3] I tread carefully here, as there are two unexploded philosophical disputes in the vicinity. First, there is a well-known dispute about whether it can happen that two things *coincide*, or are "made of the same stuff", in some sense. Second, there is a dispute about whether coincidence is properly understood as mutual parthood, or in some other way. I wish to take sides on neither of these issues: that is why I have used the term "mutual parthood" here, to mean "being part of each other", whether or not that is the right way to think about coincidence.

[4] What if a theory describes a relation as "proper part" but has neither of the biconditionals as theorems? Then the theorist has made a mistake, for the relation they are describing does not correspond to either of the senses "proper part" could have.

doesn't matter how we interpret their "proper part" relation — the two available senses of "proper part" would be equivalent in those systems.

Now that we have distinguished the two possible meanings of "proper part", let us extend the concept of a pre-mereology to include them. Let a pre-mereology be a first-order theory with at least four dyadic predicates $<$, $\circ$, $\lneq$, and $\ll$ and at least the following theorems:

$$x < y \wedge y < z \to x < z \tag{Trans}$$

$$x < x \tag{Refl}$$

$$x \circ y \leftrightarrow (\exists z)(z < x \wedge z < y) \tag{EA}$$

$$x \lneq y \leftrightarrow x < y \wedge x \neq y \tag{NIPP}$$

$$x \ll y \leftrightarrow x < y \wedge \neg y < x \tag{NMPP}$$

The simplest way to understand Simons's axiomatisation is by taking his primitive to be $\lneq$. Let us say that a theory is *Simons-axiomatisable* iff it has an axiomatisation treating $\lneq$ as primitive, and defining $<$ in the following way:

$$x < y \equiv_{df} x \lneq y \vee x = y$$

Every pre-mereology (and thus every mereology) is Simons-axiomatisable. *Proof:* for the same reasons given in the case of Goodman-axiomatisability in the previous section, a theory is Simons-axiomatisable if the biconditional corresponding to Simons's definition of $<$ is a theorem. The biconditional corresponding to the definition above is a consequence of (Refl) and (NIPP) (this proof is left to the reader). So every pre-mereology is Simons-axiomatisable.

Simons-style axiomatisations of mereologies in which the part-whole relation fails to be anti-symmetric are, however, likely to be ugly. If mutual parthood is allowed, $\lneq$ fails to satisfy most of the familiar characteristics of relations that are used as axioms by Simons. Suppose x and y are mutual parts but not identical: then $x \lneq y$ and $y \lneq x$ but not $x \lneq x$; $\lneq$ is not transitive. Similarly, if mutual parthood is allowed, $\lneq$ fails to satisfy Simons's weak supplementation principle (Cotnoir 2010, p. 399). There may be elegant Simons-style axiomatisations of mutual parthood mereologies, but there have never been any presented in the literature, and I doubt that there ever will be.

What about axiomatisations of mereologies taking $\ll$ as the primitive? To determine under what circumstances this is possible, we first have to determine what the definition of $<$ in terms of $\ll$ would be. Suppose we used Simons's definition, substituting $\ll$ for $\lneq$:

$$x < y \equiv_{df} x \ll y \vee x = y$$

There is something wrong with this definition, however. The difference between $\lneq$ and $\ll$ is that if x and y are mutually part of each other, and are distinct, then $x \lneq y$ but not $x \ll y$. If there were two such things, then this definition would wrongly count them not as mutual parts. What's needed is something more like this:

$$x < y \equiv_{df} x \ll y \vee x \text{ is mutually part of } y$$

But that is no good for we have no way of defining "is mutually part of" in terms of $\ll$.[5] Suppose we set aside these qualms, and used the Simons definition of $<$ in terms of $\ll$ shown above. A theory is axiomatisable in this way iff it has the following biconditional as a theorem:

$$x < y \leftrightarrow x \ll y \vee x = y \tag{NMPA}$$

All and only those pre-mereologies in which (NMPA) is a theorem also have this theorem — anti-symmetry:

$$x < y \wedge y < x \to x = y \tag{ASym}$$

Proof: (ASym) is a consequence of (NMPA) and (NMPP). (NMPA) is a consequence of (NMPP), (ASym) and (Refl). (NMPP) and (Refl) are theorems of every pre-mereology. So if a pre-mereology has (NMPA) as a theorem, then it has (ASym), and vice versa.

All and only those pre-mereologies in which (ASym) is a theorem also have this theorem:

$$x \ll y \leftrightarrow x \lneq y \tag{PPEq}$$

Proof: (PPEq) is a consequence of (ASym), (NMPP), and (NIPP). (ASym) is a consequence of (PPEq), (NMPP), and (NIPP). (NIPP) and (NMPP) are theorems of every pre-mereology. So if a pre-mereology has (ASym), then it has (PPEq), and vice versa.

So axiomatisations in terms of $\ll$ are only possible for mereologies in which the part-whole relation is anti-symmetric; those theories are precisely the ones in which the difference between $\lneq$ and $\ll$ doesn't matter.

[5] It won't do to say that two things are mutual parts iff they have all and only the same non-mutual parts and are non-mutually part of all and only the same — that would count all mereological atoms as mutually parts of each other. Nor will it help to say that two things are mutual parts iff they have all and only the same non-mutual parts and are non-mutually part of all and only the same and have some non-mutual part — that would make it impossible for two mereological atoms to be mutual parts of one another, which situation is intended to be possible in the types of non-extensional mereology we are dealing with here.

A First-order proofs

I here set out proofs of the sequents of first-order classical logic I relied on in the body of the paper. The proofs are in an abbreviated natural deduction notation, with some of the more tedious steps elided. I also use $|$ to mean "disjoint": $x|y$ is an abbreviation for $\neg x \circ y$.

In the proofs below, unbound letters x, y, z, w, v are to be understood as names of individuals, making it easier to follow the quantifier elimination and introduction steps. Note that this policy differs from the generality interpretation of open formulae used in the body of the paper.

A.1 (SSP) is a consequence of (GA), (Trans) and (EA)

	(1)	$\neg x < y$	assumption for conditional proof
1	(2)	$(\exists z)(z \circ x \wedge z|y)$	from (GA), 1
1	(3)	$z \circ x \wedge z|y$	$\exists$ elimination, 2
1	(4)	$(\exists w)(w < z \wedge w < x)$	from (EA), 3
1	(5)	$w < z \wedge w < x$	$\exists$ elimination, 4
1	(6)	$v < w$	assumption for conditional proof
1,6	(7)	$v < z$	from (Trans), 6
1,6	(8)	$\neg v < y$	from (EA), 7, 3
1	(9)	$v < w \rightarrow \neg v < y$	conditional proof, 8, discharging 6
1	(10)	$(\forall v)(v < w \rightarrow \neg v < y)$	$\forall$ introduction, from 9
1	(11)	$\neg(\exists v)(v < w \wedge v < y)$	from 10
1	(12)	$w|y$	from (EA), 11
1	(13)	$w < x \wedge w|y$	from 5, 12
1	(14)	$(\exists z)(z < x \wedge z|y)$	$\exists$ introduction, 13
	(15)	$\neg x < y \rightarrow (\exists z)(z < x \wedge z|y)$	conditional proof, 14, discharging 1

A.2 (GA) is a consequence of (SSP), (Trans), (Refl) and (EA).

	(1)	$x < y$	assumption for conditional proof
1	(2)	$z \circ x$	assumption for conditional proof
1,2	(3)	$(\exists w)(w < z \wedge w < x)$	from (EA), 2
1,2	(4)	$w < z \wedge w < x$	$\exists$ elimination, 3
1,2	(5)	$w < y$	from (Trans), 4
1,2	(6)	$(\exists w)(w < z \wedge w < y)$	$\exists$ introduction, 4, 5
1,2	(7)	$z \circ y$	from (EA), 6
1	(8)	$z \circ x \rightarrow z \circ y$	conditional proof, 7, discharging 2
1	(9)	$(\forall z)(z \circ x \rightarrow z \circ y)$	$\forall$ introduction, 8
	(10)	$x < y \rightarrow (\forall z)(z \circ x \rightarrow z \circ y)$	conditional proof, 9, discharging 1
	(11)	$\neg x < y$	assumption for conditional proof
11	(12)	$(\exists z)(z < x \wedge z \mid y)$	from (SSP), 11
11	(13)	$z < x \wedge z \mid y$	$\exists$ elimination, 12
11	(14)	$z \circ x$	from (EA), (Refl), 13
11	(15)	$(\exists z)(z \circ x \wedge z \mid y)$	$\exists$ introduction, 13, 14
	(16)	$\neg x < y \rightarrow (\exists z)(z \circ x \wedge z \mid y)$	conditional proof, 15, discharging 11
	(17)	$(\forall z)(z \circ x \rightarrow z \circ y) \rightarrow x < y$	contraposing 16
	(18)	$x < y \leftrightarrow (\forall z)(z \circ x \rightarrow z \circ y)$	from 10, 17

A.3 (ASym) is a consequence of (NMPP) and (NMPA)

	(1)	$x < y \wedge y < x$	assumption for conditional proof
1	(2)	$\neg x \ll y$	from (NMPP), 1
1	(3)	$x \ll y \vee x = y$	from (NMPA), 1
1	(4)	$x = y$	disjunctive syllogism, 2, 3
	(5)	$(x < y \wedge y < x) \rightarrow x = y$	conditional proof, 4, discharging 1

A.4 (NMPA) is a consequence of (NMPP), (Refl) and (ASym)

	(1)	$x < y$	assumption for conditional proof
1	(2)	$\neg y < x \vee x = y$	from (ASym),1
1	(3)	$(x < y \wedge \neg y < x) \vee x = y$	from 1, 2
1	(4)	$x \ll y \vee x = y$	from (NMPP), 3
	(5)	$x < y \rightarrow (x \ll y \vee x = y)$	conditional proof, 4, discharging 1
	(6)	$x = y \rightarrow x < y$	from (Refl)
	(7)	$x \ll y \rightarrow x < y$	from (NMPP)
	(8)	$(x \ll y \wedge x = y) \rightarrow x < y$	from 6, 7
	(9)	$x < y \leftrightarrow (x \ll y \vee x = y)$	from 5, 8

A.5 (PPEq) is a consequence of (NIPP), (NMPP) and (ASym)

	(1)	$x \lneq y$	assumption for conditional proof
1	(2)	$x < y \wedge y \neq x$	from (NIPP), 1
1	(3)	$\neg y < x$	from (ASym), 2
1	(4)	$x < y \wedge \neg y < x$	from 2, 3
1	(5)	$x \ll y$	from (NMPP), 4
	(6)	$x \lneq y \rightarrow x \ll y$	conditional proof, 5, discharging 1
	(7)	$x \ll y$	assumption for conditional proof
7	(8)	$x < y \wedge \neg y < x$	from (NMPP), 7
7	(9)	$x \neq y$	Leibniz's law, 8
7	(10)	$x < y \wedge x \neq y$	from 8, 9
7	(11)	$x \lneq y$	from (NIPP), 10
	(12)	$x \ll y \rightarrow x \lneq y$	conditional proof, 11, discharging 7
	(13)	$x \ll y \leftrightarrow x \lneq y$	from 6, 12

References

Cotnoir, A. (2010). Anti-symmetry and non-extensional mereology. *Philosophical Quarterly 60*, 396–405.

Goodman, N. (1951). *The Structure of Appearance*. Cambridge, MA: Harvard University Press.

Hovda, P. (2009). What is classical mereology? *Journal of Philosophical Logic 38*, 55–82.

Simons, P. (1987). *Parts: a Study in Ontology*. Oxford: Clarendon Press.